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Sub-Stonean Spaces and Corona Sets

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A self-contained account of the theory of sub-Stonean spaces, and their relations to Stonean spaces and Rickart spaces is given. Of particular interest are the corona sets (of the form $\beta(X)\backslash X$) for locally compact, σ-compact spaces, because these highly nontrivial sub-Stonean spaces lend themselves to Čech-cohomological considerations. The theory of sub-Stonean spaces is essential for our solution of the diagonalization problem for $C(X) \otimes \mathbb{M}_n$, found in K. Grove and G. K. Pedersen, Diagonalizing matrices over $C(X)$, submitted for publication.

INTRODUCTION

A locally compact Hausdorff space X is a *sub-Stonean space* if any two disjoint, open, σ-compact subsets of X have disjoint, compact closures. (In particular, every open, σ-compact subset is pre-compact.) By deleting the word σ-compact we are back at the definition of a Stonean space, cf. [9], but it would be misleading to think of sub-Stonean spaces merely as a generalization. In fact, sub-Stonean spaces, although elusive by nature, arise quite often in functional analysis and deserve, we think, more attention than have hitherto been allotted them.

Gillman and Henriksen studied sub-Stonean spaces (under the name of F-spaces, and in the category of completely regular spaces) in [11], see also [12, 14.25–29]. Measures on these spaces were studied in [25], and their importance in density theorems (a la Stone–Weierstrass) and interpolation theory was explained in [1; 2]. Choquet rediscovered sub-Stonean spaces, and proved in [5] (see also [6]) a number of their basic properties. In order to make our account self-contained we have allowed a certain overlap with these previous papers.

We begin with a series of somewhat elementary results about general sub-Stonean spaces. Most useful is the fact (partially reflected in the name) that closed subspaces of sub-Stonean spaces are again sub-Stonean. More striking, perhaps, is the supernatural version of Tietze's extension theorem

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that holds in sub-Stonean spaces: Any continuous function from an open, σ-compact subset into a compact space has a unique extension to the closure of the subset. The pinching and glueing technique that applies (with some care) in the category of sub-Stonean spaces is interesting also.

A special type of sub-Stonean spaces are the *Rickart spaces*, which are characterized by the condition that open, σ-compact subsets have open and compact closures. We show that they are exactly those locally compact Hausdorff spaces X for which $C_0(X)$ is monotone sequentially complete. Every Rickart space is a totally disconnected sub-Stonean space; the converse is wrong.

The most interesting sub-Stonean spaces arise as *corona sets*, of the form $\beta(X)\backslash X$ for some locally compact, σ-compact Hausdorff space X. In these spaces every open, σ-compact subset is the interior of its closure, so no corona set is a Rickart space. We show that the corona is connected whenever X is connected at infinity, and that the covering dimension of the corona is equal to the dimension of X at infinity.

For a compact set X the *real corona* $\rho(X)$ is defined as the corona of $X \times \mathbb{R}_+$. We show that the map $\pi\colon \rho(X) \to X$ obtained via the projection of $X \times \mathbb{R}_+$ on X induces an injective map from homotopy classes of maps from X to a CW-complex Z into the corresponding class for the real corona. Applied with Z an Eilenberg–MacLane space this produces a group injection

$$\pi^*\colon H^n(X, G) \to H^n(\rho(X), G)$$

of Čech cohomology groups when G is abelian. The map is not surjective in general. However, for a compact group G we are able to show that X admits a non-trivial G-bundle if and only if $\rho(X)$ does.

The authors stumbled over sub-Stonean spaces in our search for necessary and sufficient conditions for the continuous diagonalization of pointwise normal matrix-functions on a compact space X. Having solved the problem we realized that about half the work consisted of a general theory of sub-Stonean spaces, which we then decided to publish separately. The reader who wishes to know our original motivation for working with these spaces and to understand our selection of material (for surely there is more to be said about sub-Stonean spaces), must read [13].

1. General Topology

The following function-algebraic characterization of sub-Stonean spaces will be used repeatedly. Its proof is immediate once it is realized that the σ-compact, open subsets of X are precisely the co-zero sets for elements in $C_0(X)_+$ (the non-negative functions in $C_0(X)$). To see this, write $Y = \bigcup C_n$

and choose Urysohn functions f_n in $C_0(X)$ such that $f_n \mid C_n = 1$ and $f_n \mid X \setminus Y = 0$. Then with $f = \sum 2^{-n} f_n$ we have

$$Y = \{x \in X \mid f(x) > 0\}.$$

Similarly the proof of 1.2 involves only simple manipulations with the defining properties of the spaces, and is left to the reader.

PROPOSITION 1.1. *A locally compact Hausdorff space X is sub-Stonean if and only if the following condition holds: Whenever f and g in $C_0(X)$ are orthogonal (i.e., $fg = 0$) there are orthogonal functions f_1, g_1 in $C_0(X)$ such that $f_1 f = f$ and $g_1 g = g$ (local units).*

PROPOSITION 1.2. *Every image of a sub-Stonean space under a proper, open, continuous map is itself a sub-Stonean space.*

LEMMA 1.3. *If X_0 is a closed subset of a locally compact Hausdorff space X, and f_0 and g_0 are orthogonal elements in $C_0(X_0)$, then they have orthogonal extensions f, g in $C_0(X)$.*

Proof. Assume first that $f_0 \geqslant 0$ and $g_0 \geqslant 0$. Choose positive extensions f_1 and g_1 in $C_0(X)$ (Tietze) and replace these with the functions $f = (f_1 - g_1) \vee 0$ and $g = (g_1 - f_1) \vee 0$. Now $fg = 0$ in $C_0(X)$ and $f \mid X = f_0$, $g \mid X = g_0$.

In the general case write $f_0 = (f_0 |f_0|^{-1/2}) |f_0|^{1/2}$ and $g_0 = (g_0 | g_0|^{-1/2}) | g_0|^{1/2}$. Choose arbitrary extensions in $C_0(X)$ for the first factors and orthogonal extensions for the pair $|f_0|^{1/2}$, $| g_0|^{1/2}$. Multiplying the extensions together we then have orthogonal extensions of f_0 and g_0.

THEOREM 1.4 ([11, 2.2] and [5, 3]). *Every closed subset X_0 of a sub-Stonean space X is itself a sub-Stonean space.*

Proof. If $f_0 g_0 = 0$ in $C_0(X_0)$ they have orthogonal extensions $\tilde{f}_0, \tilde{g}_0$ in $C_0(X)$ by 1.3. By assumption (cf. 1.1) there are orthogonal local units $\tilde{f}_1$ and $\tilde{g}_1$ for $\tilde{f}_0$ and $\tilde{g}_0$, and their restrictions f_1 and g_1 to X_0 then provides an orthogonal pair of local units for f_0 and g_0 in $C_0(X_0)$.

PROPOSITION 1.5 [11, 2.4; 5, 4]. *An infinite sub-Stonean space contains no non-trivial convergent sequences.*

Proof. If $x_n \to x_\infty$ in the sub-Stonean space X then $X_0 = \{x_n \mid 1 \leqslant n \leqslant \infty\}$ is a closed subspace. However, the two disjoint, open, σ-compact subsets $Y = \{x_{2n-1}\}$ and $Z = \{x_{2n}\}$ have a common boundary, viz. $\{x_\infty\}$, so X_0 is not sub-Stonean.

COROLLARY 1.6. *Any first countable sub-Stonean space is finite.*

PROPOSITION 1.7 ([12, 14Q]). *If X and Y are infinite, sub-Stonean spaces then $X \times Y$ is not a sub-Stonean space.*

Proof. There exist open, σ-compact, non-compact subsets $X_0 \subset X$ and $Y_0 \subset Y$, so we can find x in $\bar{X}_0 \backslash X_0$ and y in $\bar{Y}_0 \backslash Y_0$. If $X \times Y$ was sub-Stonean the closed subset $(X \times y) \cup (x \times Y)$ would be sub-Stonean by 1.4; but it contains two disjoint, open, σ-compact subsets (viz. $X_0 \times y$ and $x \times Y_0$), with the common boundary point (x, y).

PROPOSITION 1.8. *Every continuous map $f: Y \to X$ from a connected, first countable space Y to a sub-Stonean space X is constant.*

Proof. Take x in $f(Y)$ and let $Z = f^{-1}(\{x\})$. If Z is not open there is a sequence (y_n) in $Y \backslash Z$ converging to some point in Z. But then $(f(y_n))$ is a non-trivial, convergent sequence in X, in contradiction with 1.5. Thus Z is both open and closed, whence $Z = Y$.

COROLLARY 1.9. *A sub-Stonean space X is arcwise totally disconnected ($\pi_0(X) = X$), and all homotopy groups $\pi_n(X)$, $n \geq 1$, vanish. Homotopy equivalent sub-Stonean spaces are homeomorphic.*

THEOREM 1.10. *The closure of every open, σ-compact subset Y of a sub-Stonean space X is homeomorphic to the Stone–Čech compactification $\beta(Y)$ of Y.*

Proof. Since $\bar{Y}$ is a compactification of Y, there is a continuous surjection $\varepsilon: \beta(Y) \to \bar{Y}$, extending the inclusion map of Y into $\bar{Y}$. Since $C(\beta(Y)) = C_b(Y)$, there is for each pair of distinct points z_1, z_2 in $\beta(Y)$ a bounded continuous function f on Y with $f(z_1) = 0$ and $f(z_2) = 1$. If $x \in \bar{Y}$ with $\varepsilon(z_1) = \varepsilon(z_2) = x$ this implies that both sets

$$Y_1 = \{y \in Y \mid f(y) < \tfrac{1}{2}\} \qquad \text{and} \qquad Y_2 = \{y \in Y \mid f(y) > \tfrac{1}{2}\}$$

intersect every neighbourhood of x, i.e., $x \in \bar{Y}_1 \cap \bar{Y}_2$. However, Y_1 and Y_2 are open F_σ-subsets of Y, which is open and σ-compact in X, and thus Y_1 and Y_2 are open and σ-compact in X. Since $Y_1 \cap Y_2 = \varnothing$ we have reached a contradiction. Consequently ε is injective, and thus a homeomorphism of $\beta(Y)$ onto $\bar{Y}$.

COROLLARY 1.11 ([5, 2(6)]). *Every continuous map $f: Y \to Z$ from an open, σ-compact subset Y of a sub-Stonean space into a compact Hausdorff space Z, has a (unique) continuous extension $\tilde{f}: \bar{Y} \to Z$.*

A subset S of a topological space X is *basically isolated* if $Y \cap S = \varnothing$ implies $\bar{Y} \cap \bar{S} = \varnothing$ for every open, σ-compact subset Y of X. An obvious example of a basically isolated set in a sub-Stonean space X is any set S where $Y \subset S \subset \bar{Y}$ for some open, σ-compact subset Y; but there are others. In fact, from 1.14 and 3.5 we can deduce the existence of connected sub-Stonean spaces with basically isolated points.

Note that an open subset Y of a sub-Stonean space X is sub-Stonean in the relative topology if and only if $X \setminus Y$ is basically isolated in X.

LEMMA 1.12. *If S is basically isolated in X then $(Y \cap S)^- = \bar{Y} \cap \bar{S}$ for every open, σ-compact subset Y of X.*

Proof. Clearly $(Y \cap S)^- \subset \bar{Y} \cap \bar{S}$. But if $x \notin (Y \cap S)^-$ there is an open, σ-compact neighbourhood Z of x such that $Z \cap (Y \cap S) = \varnothing$; whence by assumption $(Z \cap Y)^- \cap \bar{S} = \varnothing$. If therefore $x \in \bar{S}$ then $x \notin (Z \cap Y)^-$, whence $Z_1 \cap (Z \cap Y) = \varnothing$ for some open, σ-compact neighbourhood Z_1 of x; which means that $x \notin \bar{Y}$. Consequently $\bar{Y} \cap \bar{S} \subset (Y \cap S)^-$.

THEOREM 1.13. *Let S_1 and S_2 be basically isolated closed subsets of sub-Stonean spaces X_1 and X_2, respectively, and let $\varphi: S_1 \to S_2$ be a continuous, open and surjective map. Then the space $X = X_1 \cup_\varphi X_2$, obtained by glueing together X_1 and X_2 with φ is a sub-Stonean space.*

Proof. Clearly X with the quotient topology is a locally compact Hausdorff space. Moreover, we may identify elements f in $C_0(X)$ with pairs (f_1, f_2) in $C_0(X_1) \times C_0(X_2)$, such that $f_1 = f_2 \circ \varphi$ on S_1.

If $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are orthogonal elements in $C_0(X)$, there are orthogonal local units $\tilde{f}_1, \tilde{g}_1$ for f_1 and g_1 in $C_0(X_1)$ and orthogonal local units $\tilde{f}_2, \tilde{g}_2$ for f_2 and g_2 in $C_0(X_2)$. Put

$$Y_1 = \{x \in S_1 \mid \tilde{f}_1(x) \neq \tilde{f}_2(\varphi(x)) \text{ or } \tilde{g}_1(x) \neq \tilde{g}_2(\varphi(x))\},$$

$$Z_1 = \{x \in X_1 \mid f_1(x) \neq 0 \text{ or } g_1(x) \neq 0\},$$

$$Z_2 = \{x \in X_2 \mid f_2(x) \neq 0 \text{ or } g_2(x) \neq 0\}.$$

We claim that $\varphi(Y_1) \cap Z_2 = \varnothing$. For if $z_2 \in S_2 \cap Z_2$ we may assume that $f_2(z_2) \neq 0$. (The case $g_2(z_2) \neq 0$ is symmetric.) Then $f_1(y_1) = f_2(z_2) \neq 0$ for any y_1 in S_1, such that $\varphi(y_1) = z_2$. Consequently $\tilde{f}_1(y_1) = 1$ and $\tilde{f}_2(z_2) = 1$, whence $\tilde{g}_1(y_1) = 0$ and $\tilde{g}_2(z_2) = 0$, which shows that $y_1 \notin Y_1$.

Since φ is an open map and Y_1 is relatively open and σ-compact it follows that $\varphi(Y_1)$ is relatively open and σ-compact in S_2, and therefore of the form $Y_2 \cap S_2$ for some open, σ-compact subset Y_2 of X_2. By 1.12 the equation $Y_2 \cap S_2 \cap Z_2 = \varnothing$ implies that

$$(Y_2 \cap S_2)^- \cap \bar{Z}_2 = \bar{Y}_2 \cap S_2 \cap \bar{Z}_2 = S_2 \cap (Y_2 \cap Z_2)^- = \varnothing.$$

We can therefore find a Urysohn function h_2 which is 0 on $\varphi(Y_1)$ and 1 on Z_2. Define h_1 on $S_1 \cup \bar{Z}_1$ by $h_1 = h_2 \circ \varphi$ on S_1 and $h_1 = 1$ on $\bar{Z}_1$. This is a consistent definition, for if $x \in S_1 \cap Z_1$ then $\varphi(x) \in S_2 \cap Z_2$ whence $h_2(\varphi(x)) = 1$; and by 1.12 we know that $S_1 \cap \bar{Z}_1 = (S_1 \cap Z_1)^-$. Extend h_1 to all of X_1 (Tietze) and define the pairs

$$\tilde{f} = (h_1 \tilde{f}_1, h_2 \tilde{f}_2) \qquad \text{and} \qquad \tilde{g} = (h_1 \tilde{g}_1, h_2 \tilde{g}_2).$$

By definition $(h_1, h_2) \in C_0(X)$ and by construction $\tilde{f} \in C_0(X)$ and $\tilde{g} \in C_0(X)$. Of course $\tilde{f}\tilde{g} = 0$, so it remains to show that they are local units for f and g. But if $x \in X_1$ with $f(x_1) \neq 0$, then $\tilde{f}_1(x) = 1$ and also $h_1(x) = 1$ since $x \in Z_1$. The other three cases are analogous and the proof is complete.

COROLLARY 1.14. *If S is a basically isolated, closed subset of a sub-Stonean space X, then the space obtained by contracting S to a point is a sub-Stonean space with a basically isolated point.*

PROPOSITION 1.15. *Every sub-Stonean space which is the image of a product of compact metric spaces under an open continuous map is a Stonean space.*

Proof. If X is a product of compact metric spaces then by Bockstein's separation result [4], any two disjoint open subsets of X have disjoint open, σ-compact neighbourhoods. Obviously this result also holds when X is the image of a product space. If now X is a sub-Stonean space we see that any two disjoint open subsets of X have disjoint closures. Applied to an open subset Y and $X \backslash \bar{Y}$ it shows that $\bar{Y} \cap (X \backslash \bar{Y})^- = \varnothing$, i.e., $\bar{Y}$ is open.

PROPOSITION 1.16. *Any compact, sub-Stonean group is finite.*

Proof. By a theorem of Kuzminov, see, e.g., [23, 7.6], every compact group G is the quotient of a product of compact metric groups. By 1.15 we know then that G is a Stonean space. If G was infinite, the arguments on p. 169 of [7] show that there is a closed subgroup H of G, such that G/H is infinite, but separable. Since G/H is also Stonean (open, disjoint sets have disjoint closures) this is impossible. Thus G is finite.

2. RICKART SPACES

Following [22, 4.5.5] we say that a C^*-algebra $\mathfrak{A}$ of operators on some Hilbert space H is a *Borel *-algebra* if $\mathfrak{A}_{sa}$ (the self-adjoint part of $\mathfrak{A}$) is monotone sequentially closed in $\mathbb{B}(H)$. To obtain a Hilbert space free description we define a *monotone σ-complete C^*-algebra* to be a C^*-algebra

$\mathfrak{A}$ in which each bounded, monotone increasing sequence in $\mathfrak{A}_{sa}$ has a least upper bound in $\mathfrak{A}_{sa}$. We further say that a positive functional φ on $\mathfrak{A}$ is *σ-normal* if $\lim \varphi(A_n) = \varphi(A)$, whenever A is the least upper bound of the increasing sequence (A_n) in $\mathfrak{A}_{sa}$. It now follows from the Gelfand–Naimark–Segal construction (combine [22, 3.3.9 and 3.7.4]), that a C^*-algebra $\mathfrak{A}$ is a Borel $*$-algebra if and only if $\mathfrak{A}$ is monotone σ-complete and has a separating family of σ-normal functionals.

We say that a locally compact Hausdorff space X is a *Rickart space* if the closure of every σ-compact open subset of X is open and compact. Clearly this is equivalent to the conditions (preferred in [3]) that the compact-open subsets of X form a basis for the topology and is a σ-lattice (with $\bigvee Y_n = (\bigcup Y_n)^-$). In [12, Problem 1H] these spaces are said to be basically disconnected. If further X has a family of Radon measures μ, such that $\mu(\overline{Y}\backslash Y) = 0$ for every σ-compact, open subset Y in X, and $\mu_0(Y) > 0$ for at least some μ_0 in the family, then X is called a *hyper Rickart space*. Clearly every Rickart space is a totally disconnected sub-Stonean space, and both terms (Rickart and sub-Stonean) are possible sequential versions of Stonean spaces.

The fact, spelled out in the next proposition, that the Rickart spaces relate to the monotone σ-complete C^*-algebras exactly as the Stonean spaces relate to AW^*-algebras is due (in essence) to Rickart [24], and the proof in [3, Sect. 8, Theorem 1] can easily be modified to give the result. For the convenience of the reader we include a direct proof.

THEOREM 2.1. *The commutative C^*-algebra $C_0(X)$ is monotone σ-complete (respectively a Borel $*$-algebra) if and only if X is a Rickart space (resp. a hyper Rickart space).*

Proof. Assume first that $C_0(X)$ is monotone σ-complete, and let Y be a σ-compact open subset of X. From Urysohn's lemma we obtain a sequence (f_n) in $C_0(X)$ vanishing on $X\backslash Y$, such that $Y = \bigcup\{x \in X \mid f_n(x) = 1\}$. We may evidently assume that the sequence (f_n) is increasing, and by assumption it therefore has a least upper bound f in $C_0(X)$. Since $f^2 \leqslant f$ and $Y \subset \{x \in X \mid f(x) = 1\}$, we conclude from the uniqueness of f that $f^2 = f$, i.e., f is the characteristic function for an open and compact set Z containing Y. If $\overline{Y} \neq Z$ there is an open set $A \subset Z\backslash \overline{Y}$, and thus a non-zero function g in $C_0(X)$, $0 \leqslant g \leqslant 1$, with support in A. But then $f - g$ is a continuous majorant for (f_n) smaller than f, a contradiction.

If X is a Rickart space and (f_n) is a bounded monotone increasing sequence in $C_0(X)$, we may evidently assume that $0 \leqslant f_n \leqslant 1$ for all n. Put $g = \bigvee f_n$ (as functions on X) and for a fixed m in $\mathbb{N}$ let

$$Y_k^m = \{x \in X \mid g(x) > (k-1)\, 2^{-m}\}^-, \qquad 1 \leqslant k \leqslant 2^m.$$

Since for each $t \geqslant 0$,

$$\{x \in X \mid g(x) > t\} = \bigcup \{x \in X \mid f_n(x) > t\}$$

$$= \bigcup \{x \in X \mid f_n(x) \geqslant t + n^{-1}\},$$

we see from the assumption that each Y_k^m is open and compact, and thus the characteristic set for a projection g_k^m in $C_0(X)$. Thus

$$g_m = 2^{-m} \sum_{k=1}^{2^m} g_k^m \in C_0(X).$$

For each x in X with $g(x) > 0$ there is a unique k such that

$$(k-1)\, 2^{-m} < g(x) \leqslant k2^{-m}.$$

This means that $x \in Y_j^m$ for all $j \leqslant k$, whence

$$g(x) \leqslant k2^{-m} \leqslant g_m(x).$$

On the other hand $Y_{2k+1}^{m+1} = Y_{k+1}^m$, so if $g_m(x) = k2^{-m}$, i.e., $x \in Y_k^m \backslash y_{k+1}^m$, then $x \in y_{2k-1}^{m+1} \backslash y_{2k+1}^{m+1}$, so that either $g_{m+1}(x) = (2k-1)\, 2^{-m-1}$ or $g_{m+1}(x) = 2k2^{-m-1}$. Consequently,

$$g_m(x) - 2^{-m} \leqslant g_{m+1}(x) \leqslant g_m(x).$$

The sequence (g_m) is therefore uniformly convergent to a continuous function $f \geqslant g$, i.e., a majorant in $C_0(X)$ for (f_n).

Suppose now that h is another majorant in $C_0(X)$ for (f_n). Then $g \leqslant h$, so if $(k-1)\, 2^{-m} < g(x)$ then $(k-1)\, 2^{-m} \leqslant h(x)$, whence

$$Y_k^m \subset \{x \in X \mid (k-1)\, 2^{-m} \leqslant h(x)\}.$$

If therefore $x \in Y_k^m \backslash Y_{k+1}^m$ then

$$g_m(x) = k2^{-m} \leqslant h(x) + 2^{-m}.$$

In the limit we obtain $f \leqslant h$, so that f is the smallest majorant for (f_n) in $C_0(X)$, which is therefore monotone σ-complete.

To see that hyper Rickart spaces correspond to Borel *-algebras merely amounts to show that the probability measures μ on X that satisfy $\mu(\bar{Y} \backslash Y) = 0$ for every open, σ-compact subset Y, are precisely those measures for which

$$\lim \int f_n \, d\mu = \int f \, d\mu.$$

whenever f is the least upper bound in $C_0(X)$ for the increasing sequence (f_n). But if we always have $\mu(\bar{Y}) = \mu(Y)$, then with notations as in the argument above

$$\lim \int f_n \, d\mu = \int g \, d\mu$$

$$\geqslant 2^{-m} \sum_{k>1} \mu\{x \in X \mid g(x) > (k-1) \, 2^{-m}\}$$

$$= 2^{-m} \sum_{k>1} \mu(Y_k^m) \geqslant \int g_m \, d\mu - 2^{-m} \to \int f \, d\mu,$$

as desired.

From 2.1 we obtain a large collection of sub-Stonean spaces: Take any locally compact Hausdorff space Y and let $\mathscr{B}(Y)$ denote the set of bounded Baire functions on Y. Then $\mathscr{B}(Y)$ is the enveloping Borel *-algebra of $C_0(Y)$ (cf. [22, 4.5.14]) and thus the locally compact Hausdorff space X for which $C_0(X) = \mathscr{B}(Y)$ is a hyper Rickart space (which by the way contains the original set Y as a dense, discrete subset).

It is easy to propose examples of sub-Stonean spaces that are not Rickart spaces: We just have to imagine a compact space X such that $C(X)$ is not monotone σ-complete, but is a quotient of a monotone σ-complete algebra. The first example that comes to mind is $X = \beta(\mathbb{N}) \backslash \mathbb{N}$, where $C(X) = l^\infty / c_0$. Already in 1952 I. Kaplansky noted that X is not a Stonean space (because it contains an uncountable family of pairwise disjoint clopen sets). That X also fails to be a Rickart space in the rather spectacular manner shown in 3.3 was proved to us by E. T. Kehlet. Inspection of the argument showed that this behaviour is common for all corona sets of locally compact, σ-compact Hausdorff spaces, see 3.3 and 3.4.

PROBLEM 2.2. Characterize those sub-Stonean spaces that are closed subsets of Rickart spaces.

A partial solution (covering most cases of interest) was given by Louveau in [18], assuming the continuum hypothesis: Every totally disconnected sub-Stonean space with a basis at most the cardinality of the continuum is homeomorphic to a closed subset of $\beta(\mathbb{N})$.

3. CORONA SETS

If X is a locally compact Hausdorff space with Stone–Čech compactification $\beta(X)$ we define the *corona* of X as $\chi(X) = \beta(X) \backslash X$. Thus $\chi(X)$ is a compact Hausdorff space (since X is open in $\beta(X)$) and, as we shall see, these corona sets give a whole new series of examples of sub-Stonean spaces.

Recall that a continuous map $\varphi: X \to Y$ between locally compact Hausdorff spaces is *proper* if $\varphi^{-1}(C)$ is compact in X for every compact C in Y.

PROPOSITION 3.1. *Every proper continuous map $\varphi: X \to Y$ between locally compact, σ-compact Hausdorff spaces X and Y induces, via its canonical extension $\beta\varphi: \beta(X) \to \beta(Y)$, a continuous map $\chi\varphi: \chi(X) \to \chi(Y)$, and the image of $\chi(X)$ is basically isolated in $\chi(Y)$.*

Proof. If (x_λ) is a net in X and the net $(\varphi(x_\lambda))$ converges in Y, there is a compact set C such that $\varphi(x_\lambda) \in C$ eventually. This means that $x_\lambda \in \varphi^{-1}(C)$ eventually, and thus every limit point of (x_λ) lies in X. Consequently $\chi\varphi = \beta\varphi \mid \chi(X)$ maps $\chi(X)$ into $\chi(Y)$.

The argument above also shows that $S = \varphi(X)$ is closed in Y. Since $\beta\varphi$ maps $\beta(X)$ onto the closure of S in $\beta(Y)$, which is homeomorphic to $\beta(S)$, we must show that $\chi(S)$ is basically isolated in $\chi(Y)$. Take any continuous function f on $\chi(Y)$ that is 0 on $\chi(S)$. There is then an f_1 in $C_b(Y)$ with $\beta f_1 = f$, which means that $f_1 \mid S \in C_0(S)$. Using the σ-compactness of Y we can therefore find an open set $Z \supset S$ such that $f_1 \mid Z \in C_0(Z)$. Since Y is normal there is a continuous function g_1 on Y which is 1 on S and 0 on $Y \backslash Z$. Thus $f_1 g_1 \in C_0(Y)$ so if $g = \chi g_1$ then $fg = 0$ in $C(\chi(Y))$ and g is 1 on $\chi(S)$. Consequently,

$$\chi(S) \subset \{x \in \chi(Y) \mid g(x) \neq 0\} \subset \{x \in \chi(Y) \mid f(x) = 0\},$$

which shows that f vanishes on a neighbourhood of $\chi(S)$. Thus the co-zero set of f has no limit points in $\chi(S)$, which is therefore basically isolated.

THEOREM 3.2 ([11, 2.7]). *If X is a locally compact, σ-compact Hausdorff space then its corona $\chi(X)$ is a sub-Stonean space.*

Proof. If $fg = 0$ in $C(\chi(X))$ we choose orthogonal extensions f_0, g_0 in $C(\beta(X))$ by 1.3. Set

$$A_n = \{x \in X \,\big|\, |f_0(x)| \geqslant n^{-1}\}, \qquad B_n = \{x \in X \,\big|\, |g_0(x)| \geqslant n^{-1}\}.$$

By assumption there is an increasing sequence (X_n) of open, pre-compact subsets with $\bigcup X_n = X$. Set

$$A = \bigcup A_n \backslash X_n, \qquad B = \bigcup B_n \backslash X_n.$$

Since $A \cap X_n \subset A_{n-1}$ we see that A (and similarly B) is closed in X. As X is normal and $A \cap B = \varnothing$ there are orthogonal continuous functions f_1 and g_1

in $C_b(X)$ such that f_1 is 1 on A and g_1 is 1 on B. Note now that if $x \notin X_n$ then either $x \notin A_n$, whence $|f_0(x)| < n^{-1}$, or $x \in A$, whence $f_1(x) = 1$. It follows that $(1 - f_1) f_0 \in C_0(X)$, and similarly $(1 - g_1) g_0 \in C_0(X)$, so that χf_1 and χg_1 is an orthogonal pair of local units for f and g.

THEOREM 3.3. *If X is a locally compact, σ-compact Hausdorff space then every open, σ-compact subset of its corona $\chi(X)$ is the interior of its closure.*

Proof. If Y_0 is open and σ-compact in $\chi(X)$, choose an open and σ-compact subset Y of $\beta(X)$ with $Y \cap \chi(X) = Y_0$. Take increasing sequences (B_n) and (C_n) of compact subsets of $\beta(X)$ with $\bigcup B_n = X$ and $\bigcup C_n = Y$, and without loss of generality assume that $B_n \subset \mathring{B}_{n+1}$ and $C_n \subset \mathring{C}_{n+1}$ for all n.

The theorem is trivially true if $Y_0 = \chi(X)$, so we assume $Y_0 \neq \chi(X)$ and claim that $\bar{Y}_0 \neq \chi(X)$. To prove this, note that if

$$(X \backslash B_n) \cap (X \backslash C_m) = \varnothing$$

for some n, m, then $X \subset B_n \cup C_m$, whence $\chi(X) \subset C_m \subset Y$, in contradiction with $\chi(X) \neq Y_0$. Passing if necessary to subsequences of (B_n) and (C_m), we can find a sequence (A_n) of non-empty, compact G_δ-subsets of X with

$$A_n \subset (\mathring{B}_{n+1} \backslash B_n) \cap (\mathring{C}_{n+1} \backslash C_n).$$

Put $A = \bigcup A_n$ and note that A is closed in X, since

$$A \cap \mathring{B}_{n+1} \subset A_1 \cup \cdots \cup A_n.$$

However, A is not compact in X because the A_n's are pairwise disjoint. The equations

$$C_n \backslash A = C_n \backslash (A_1 \cup \cdots \cup A_{n-1}),$$
$$\mathring{C}_n \backslash A = \mathring{C}_n \backslash (A_1 \cup \cdots \cup A_{n-1}),$$

Show that each $C_n \backslash A$ is σ-compact (as the A_k's are G_δ-subsets of C_{k+1}) and each $\mathring{C}_n \backslash A$ is open in $\beta(X)$. Consequently, $Y \backslash A$ is an open, σ-compact subset of $\beta(X)$ and $(Y \backslash A) \cap \chi(X) = Y_0$. Since the inclusion map $A \to X$ is proper, it follows from 3.1 that

$$\chi(A) \subset \bar{A} \cap \chi(X) \subset \chi(X) \backslash (Y \backslash A) = \chi(X) \backslash Y_0.$$

Thus $\chi(A) \cap Y_0 = \varnothing$, and since $\chi(A)$ is non-empty and basically isolated in $\chi(X)$ we see that $\bar{Y}_0 \neq \chi(X)$.

If C is a compact G_δ-set in $\chi(X)$ contained in $\bar{Y}_0$, then $(\chi(X) \backslash C) \cup Y_0$ is an open, σ-compact subset which is dense in $\chi(X)$. As we saw above, this implies that $(\chi(X)\backslash C) \cup Y_0 = \chi(X)$, i.e., $C \subset Y_0$. Since every interior point in $\bar{Y}_0$ has a neighbourhood $C \subset \bar{Y}_0$ which is a compact G_δ-set, we see that Y_0 is the interior of $\bar{Y}_0$.

COROLLARY 3.4. *No corona set is a Rickart space.*

Proof. If $\chi(X)$ was a Rickart space for some locally compact, σ-compact (non-compact) Hausdorff space X, then for every open, σ-compact subset Y of $\chi(X)$ we have $Y = \mathring{\bar{Y}} = \bar{Y}$. However, X contains a sequence of pairwise disjoint, closed non-compact subsets, so $\chi(X)$ is infinite by 3.1. Therefore it contains a sequence (Y_n) of pairwise disjoint, open, σ-compact sets. The set $Y = \bigcup Y_n$ is open and σ-compact, but it is certainly not compact if each Y_n is also compact.

PROPOSITION 3.5 [11, 2.8]. *If X is a locally compact Hausdorff space which is connected at infinity then the corona $\chi(X)$ is connected.*

Proof. Assume, to obtain a contradiction, that there are closed sets E and F in $\chi(X)$ (therefore closed in $\beta(X)$) such that $E \cap F = \varnothing$ and $E \cup F = \chi(X)$. We can then find open disjoint subsets Y and Z in $\beta(X)$ with $E \subset Y$ and $F \subset Z$. Now $C = \beta(X)\backslash(Y \cup Z)$ is a compact subset of $\beta(X)$, and therefore compact in X, since $\chi(X) \subset Y \cup Z$. By assumption there is a larger compact set C_1 in X such that $X\backslash C_1$ is connected. However, this is impossible since we have the disjoint union

$$(Y\backslash C_1) \cup (Z\backslash C_1) \supset X\backslash C_1.$$

For a locally compact space X we define the (covering) *dimension at infinity* as

$$\dim(X, \infty) = \inf \dim(X\backslash Y),$$

where Y ranges over all open, pre-compact subsets of X.

THEOREM 3.6. *If X is a σ-compact, locally compact Hausdorff space with corona $\chi(X)$ then*

$$\dim(\chi(X)) = \dim(X, \infty).$$

Proof. By [21, 9.5], $\dim(\beta(Z)) = \dim Z$ for any normal space Z, and since the corona is independent of local phenomena we have

$$\begin{aligned} \dim(\chi(X)) &= \dim(\chi(X\backslash Y)) \\ &\leqslant \dim(\beta(X\backslash Y)) = \dim(X\backslash Y), \end{aligned}$$

for every pre-compact set Y, so that $\dim(\chi(X)) \leqslant \dim(X, \infty)$.

To prove the converse, choose a sequence (C_n) of compact subsets of X with $\bigcup C_n = X$, and without loss of generality assume that $C_n \subset \mathring{C}_{n+1}$ for every n. If $\dim(X, \infty) \geqslant d$ we may further assume that $\dim(X \backslash \mathring{C}_n) \geqslant d$ for all n, deleting if necessary a finite number. Since $X \backslash \mathring{C}_n = \bigcup C_m \backslash \mathring{C}_n$ it follows from the sum theorem [21, 9.10] that $\dim(C_m \backslash \mathring{C}_n) \geqslant d$ for some m. Passing if necessary to a subsequence we may therefore assume that $\dim(C_{n+1} \backslash \mathring{C}_n) \geqslant d$ for all n. With $I = [-1, 1]$ we can thus find a continuous function $f_n: C_{n+1} \backslash \mathring{C}_n \to I^d$ having 0 as a stable value. Using Tietze's theorem we extend f_n to X so that it vanishes off $\mathring{C}_{n+2} \backslash C_{n-1}$. This implies that f_n and f_m have disjoint supports when $|n-m| > 2$, so we may define $f = \sum f_{3n}$ to obtain a continuous function from X into I^d. Let f also denote the extension of f to $\beta(X)$.

If $\dim(\chi(X)) < d$ there is for every $\varepsilon > 0$ a continuous function $g: \chi(X) \to I^d$ such that

$$\|f \mid \chi(X) - g\| < \varepsilon \qquad \text{and} \qquad 0 \notin g(\chi(X)).$$

Let g also denote an extension from $\chi(X)$ to $\beta(X)$. There is then an open subset Y of $\beta(X)$ containing $\chi(X)$ such that

$$\|(f-g) \mid Y\| < \varepsilon \qquad \text{and} \qquad 0 \notin g(Y).$$

However, $X \backslash Y = \beta(X) \backslash Y$ is compact, so $X \backslash Y \subset C_n$ for n large enough, which means that $C_{n+1} \backslash \mathring{C}_n \subset Y$. As $f \mid C_{n+1} \backslash \mathring{C}_n = f_n$ and 0 is a stable value for f_n we have reached a contradiction. Therefore $\dim(\chi(X)) \geqslant d$ and consequently $\dim(X, \infty) \leqslant \dim \chi(X)$.

From the results in this section we see that for each $n \geqslant 1$ there is a connected sub-Stonean space of dimension n. For $n = 1$ take $X = \chi(\mathbb{R}_+)$ and for $n > 1$ take $X = \chi(\mathbb{R}^n)$ or, more economically, $X = \chi(I^{n-1} \times \mathbb{R}_+)$. For zero-dimensional examples, any totally disconnected, compact sub-Stonean space will do [21, 8.6].

Having connected sub-Stonean spaces at our disposal, the constructions in 1.13 and 1.14 become more interesting. Take for example two disjoint open, σ-compact subsets Y_1 and Y_2 of a connected sub-Stonean space X, and glue together two copies of X by identifying $\overline{Y}_1$ and $\overline{Y}_2$ (pairwise). By 1.13 the resulting connected space is sub-Stonean, but it is not unicoherent: The two copies of X are both connected, but their intersection is the disjoint union of $\overline{Y}_1$ with $\overline{Y}_2$. In some sense we have constructed a space with a loop, the result in 1.9 notwithstanding. A very canonical construction of such spaces is given by

Proposition 3.7. *If X_1 and X_2 are locally compact, σ-compact Hausdorff spaces and $\varphi: X_0 \to X_2$ is a proper continuous map on a closed subset X_0 of X_1, then*

$$\chi(X_1 \cup_\varphi X_2) = \chi(X_1) \cup_{\chi\varphi} \chi(X_2).$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
C_0(X_1 \cup_\varphi X_2) & \xrightarrow{\iota} & C_0(X_1) \oplus C_0(X_2) \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\
C_b(X_1 \cup_\varphi X_2) & \xrightarrow{\iota} & C_b(X_1) \oplus C_b(X_2) \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\
C(\chi(X_1 \cup_\varphi X_2)) & \xrightarrow{\iota} & C(\chi(X_1)) \oplus C(\chi(X_2))
\end{array}$$

Here the map ι in the third row is constructed from the others, i.e., $\iota(f) = q(\iota(g))$ for any g in $C_b(X_1 \cup_\varphi X_2)$ with $q(g) = f$. It follows that also this ι is injective and that its image lies in the set of functions (f_1, f_2) on $\chi(X_1) \times \chi(X_2)$ such that $f_1 \mid \chi(X_0) = f_2 \circ \chi\varphi$. But these coherent pairs are exactly the continuous functions on the space $\chi(X_1) \cup_{\chi\varphi} \chi(X_2)$, which is well defined by 3.1.

To see that ι is surjective, take a coherent pair (f_1, f_2) in $C(\chi(X_1)) \oplus C(\chi(X_2))$ and choose extensions g_1 and g_2 in $C(\beta(X_1))$ and $C(\beta(X_2))$, respectively. Then

$$g_1 \mid X_0 - g_2 \circ \varphi \in C_0(X_0).$$

Since X_0 is closed in X_1 this map has a continuous extension g_0 in $C_0(X_1)$. If therefore $\tilde{g}_1 = g_1 - g_0$ then $(\tilde{g}_1, g_2)$ is a coherent pair in $C_b(X_1) \oplus C_b(X_2)$ and thus corresponds to an element g in $C_b(X_1 \cup_\varphi X_2)$ with $\iota(q(g)) = (f_1, f_2)$.

We have shown that the algebras $C(\chi(X_1 \cup_\varphi X_2))$ and $C(\chi(X_1) \cup_{\chi\varphi} \chi(X_2))$ are isomorphic, and consequently the spaces $\chi(X_1 \cup_\varphi X_2)$ and $\chi(X_1) \cup_{\chi\varphi} \chi(X_2)$ are homeomorphic. In particular the second space is sub-Stonean although $\chi\varphi$ has not been proved open, so that 1.13 is not directly applicable.

4. Algebraic Topology

Let X be a compact Hausdorff space and Z a topological space whose homotopy type is that of a CW-complex. By a folklore theorem (proved for good measure in [14, 6.6]) this is equivalent to Z having the homotopy type of an ANR (absolute neighbourhood retract). We denote by $[X, Z]$ the set of homotopy classes $[f]$ of continuous maps $f: X \to Z$.

To X we associate its *real corona*

$$\rho(X) = \chi(X \times \mathbb{R}_+) = \beta(X \times \mathbb{R}_+) \backslash X \times \mathbb{R}_+.$$

By 3.2–3.4 the real corona is a sub-Stonean space which is connected whenever X is, and whose dimension is $\dim(X)+1$. One may think of $\rho(X)$ as the set of possible ways of going to infinity along $\mathbb{R}_+$ through X.

The projection $\pi\colon X\times\mathbb{R}_+\to X$ extends continuously to $\beta(X\times\mathbb{R}_+)$ and restricts to a continuous map $\pi\colon\rho(X)\to X$.

THEOREM 4.1. *The induced map*

$$\pi^*\colon [X,Z]\to[\rho(X),Z]$$

is injective.

Proof. Without loss of generality let Z be an ANR. Given f and g in $C(X,Z)$, assume that $\pi^*([f])=\pi^*([g])$. This means that we have a homotopy $H\colon\rho(X)\times[0,1]\to Z$ with $H_0=f\circ\pi$ and $H_1=g\circ\pi$. Together with f and g this defines a continuous map from the closed subset

$$F=(X\times\mathbb{R}_+\times\{0\})\cup(\rho(X)\times[0,1])\cup(X\times\mathbb{R}_+\times\{1\})$$

of $\beta(X\times\mathbb{R}_+)\times[0,1]$ into Z. Since Z is an ANR we can extend this map continuously to a closed neighbourhood of F whose complement U without loss of generality has the form

$$U=X\times[0,t[\times]\varepsilon,1-\varepsilon[$$

for some large t in $\mathbb{R}_+$ and $\varepsilon>0$. Applying the retraction of Fig. 1 we obtain a continuous extension

$$\tilde{H}\colon\beta(X\times\mathbb{R}_+)\times[0,1]\to Z$$

of H such that $\tilde{H}_0\mid X\times\{0\}=f$ and $\tilde{H}_1\mid X\times\{1\}=g$. The restriction of $\tilde{H}$ to $X\times\{0\}\times[0,1]$ is then a homotopy from f to g.

Remark 4.2. By 3.1 the corona construction χ is a covariant functor from the category of locally compact, σ-compact Hausdorff spaces with proper continuous maps to the category of compact sub-Stonean spaces with

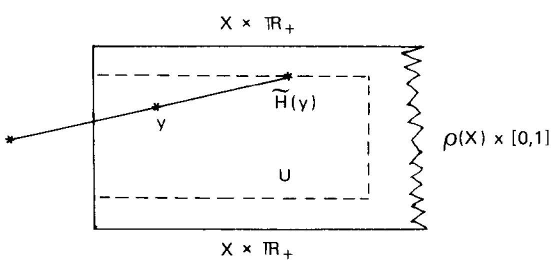

FIGURE 1

continuous maps. It follows that the real corona construction ρ is a covariant functor from the category of compact spaces to the category of compact sub-Stonean spaces.

If X is a compact CW-complex, the functor ρ transforms the cellular structure in X to $\rho(X)$ by 3.7; but note that the "n-cells" in $\rho(X)$ are real coronas of n-balls and in particular have dimension $n+1$ by 3.6. Still it might be possible to get more information about $[\rho(X), Z]$ than that contained in 4.1, by calculating homotopy classes of maps from real coronas of n-balls to Z. As Example 4.6 shows, such spaces can be huge.

Let now G be a topological group and consider the class of bundles (B, p, X) over X with structure group G. From the classification theory of G-bundles (see, e.g., [17, 1.4.12]) there is a classifying space BG which has the homotopy type of a CW-complex (or an ANR) such that the set of isomorphism classes of principal G-bundles over X is isomorphic to $[X, BG]$. From 4.1 we immediately deduce

COROLLARY 4.3. *If a compact Hausdorff space X admits a non-trivial G-bundle, then so does its real corona $\rho(X)$.*

THEOREM 4.4. *If G is a compact group then a compact Hausdorff space X admits a non-trivial G-bundle if and only if its real corona $\rho(X)$ does so.*

Proof. Using the classifying space BG, the content of the theorem is that $[\rho(X), BG]$ is trivial whenever $[X, BG]$ is. Take therefore f in $C(\rho(X), BG)$. As in the proof of 4.1 we can extend f to an $\tilde{f}$ in $C(\beta(X \times \mathbb{R}_+), BG)$ by extending first to a neighbourhood of $\rho(X)$, whose complement has the form $X \times [0, t]$, and then extending it over the complement by setting $\tilde{f}(x, s) = \tilde{f}(x, t)$ for all $s < t$.

Let EG be the total space of the classifying principal G-bundle $G \to EG \to BG$, and recall that EG is contractible (by universality) [8, 10]. Since $[X, BG]$ is trivial the map $f_0 = \tilde{f} \mid X \times \{0\}$ in $C(X, BG)$ can be lifted to g_0 in $C(X, EG)$. Applying the homotopy lifting property of the fibration $p: EG \to BG$, [27], it follows that we can lift $\tilde{f}$ in $C(X \times \mathbb{R}_+, BG)$ to $\tilde{g}$ in $C(X \times \mathbb{R}_+, EG)$. Note now that since $\beta(X \times \mathbb{R}_+)$ is compact the image $\tilde{f}(X \times \mathbb{R}_+)$ is pre-compact in BG. Since also G is compact the image of the lifted map, $\tilde{g}(X \times \mathbb{R}_+)$, is pre-compact in EG. Therefore we have a continuous extension of $\tilde{g}$ to $\beta(X \times \mathbb{R}_+)$. Set $g = \tilde{g} \mid \rho(X)$ and note that $g \circ p = f$ since $\tilde{g} \circ p = \tilde{f}$. Since EG is contractible, f is null-homotopic.

As usual, for G discrete let $K(G, n)$ denote the Eilenberg–MacLane space of type (G, n); i.e., $K(G, n)$ has the homotopy type of a CW-complex with

$$\pi_k(K(G, n)) = 0 \quad \text{for} \quad k \neq n, \qquad \pi_n(K(G, n)) = G.$$

When G is abelian, the H-space structure on $K(G, n)$ induces a group structure on $[X, K(G, n)]$, see, e.g., [20], and there is a group isomorphism

$$[X, K(G, n)] \cong H^n(X, G),$$

where $H^n(X, G)$ is the nth Čech cohomology group of X with coefficients in G. This follows from [16] but can also be deduced from [27] by the equations

$$\begin{aligned}[X, K(G, n)] &= \lim [K(\mathscr{U}), K(G, n)] \\ &= \lim H^n(K(\mathscr{U}), G) = H^n(X, G),\end{aligned}$$

where $\mathscr{U}$ ranges over the finite open coverings of X and $K(\mathscr{U})$ is the nerve of $\mathscr{U}$. The three equality signs are found in [27, 3.G.6, 8.1.10 and 6.D.3].

When G is not abelian and $n = 1$ (necessarily), then $[X, K(G, 1)]$ is not a group in general. But since $K(G, 1)$ is a classifying space for principal G-bundles, this set is isomorphic to the set of equivalence classes of principal G-bundles over X. These on the other hand are canonically identified with the sheaf cohomology set $H^1(X, \mathscr{G})$, where $\mathscr{G}$ is the sheaf associated to G, i.e., the group of local sections of $\mathscr{G}$ is $C(U, G)$, $U \subset X$, see, e.g., [15, 3.2.1]. For the relations between the two view-points of classifying principal bundles, see [26]. A survey of the classification theory of bundles is given in [28].

Allowing ourselves to use the notation $H^1(X, G)$ instead of $H^1(X, \mathscr{G})$ when G is not abelian, we deduce from 4.1,

COROLLARY 4.5. *For a compact Hausdorff space X with real corona $\rho(X)$ the map $\pi\colon \rho(X) \to X$ induces an injection*

$$\pi^*\colon H^n(X, G) \to H^n(\rho(X), G)$$

which when G is abelian is a group injection of Čech cohomology groups.

As the following example shows, the map π^* in 4.5 is in general far from being surjective.

EXAMPLE 4.6. Take X connected, and let $G = \mathbb{Z}$ and $n = 1$. Then $K(\mathbb{Z}, 1) = S^1$. We claim that

$$H^1(\rho(X), \mathbb{Z}) = C(X \times \mathbb{R}_+)/C_b(X \times \mathbb{R}_+).$$

To prove this, fix a point x_0 in X and let $C^*(X \times \mathbb{R}_+)$ (resp. $C^*(X \times \mathbb{R}_+, S^1)$) denote those continuous real (resp. S^1-valued) functions f

for which $f(x_0, 0) = 0$ (resp. $f(x_0, 0) = 1$). Consider the commutative diagram

$$\begin{array}{ccccc}
C_0^*(X \times \mathbb{R}_+) & \longrightarrow & C_b^*(X \times \mathbb{R}_+) & \longrightarrow & C^*(X \times \mathbb{R}_+) \\
\downarrow \exp & & \downarrow \exp & & \downarrow \exp \\
C_0^*(X \times \mathbb{R}_+, S^1) & \longrightarrow & C_l^*(X \times \mathbb{R}_+, S^1) & \longrightarrow & C^*(X \times \mathbb{R}_+, S^1).
\end{array}$$

Horizontally the diagram consists of injections, vertically of isomorphisms because X is connected. The class $C_l^*(X \times \mathbb{R}_+, S^1)$ consists of those functions that admit a bounded lifting to $\mathbb{R}$. Taking quotient groups with the first column we have the diagram

$$\begin{array}{ccc}
C(\rho(X)) & \longrightarrow & C(X \times \mathbb{R}_+)/C_0(X \times \mathbb{R}_+) \\
\downarrow & & \downarrow \\
C_l(\rho(X), S^1) & \longrightarrow & C(\rho(X), S^1).
\end{array}$$

Here $C_l(\rho(X), S^1)$ is precisely the class of functions on $\rho(X)$ which admit a lifting to $\mathbb{R}$. Since $\mathbb{R}$ is the universal covering space for S^1 a function f in $C(\rho(X), S^1)$ is null-homotopic if and only if it belongs to $C_l(\rho(X), S^1)$. Consequently

$$\begin{aligned}
H^1(\rho(X), \mathbb{Z}) &= [\rho(X), S^1] \\
&= C(\rho(X), S^1)/C_l(\rho(X), S^1) = C(X \times \mathbb{R}_+)/C_b(X \times \mathbb{R}_+),
\end{aligned}$$

as desired.

Taking $X = \{pt\}$ we see the discrepancy: $H^1(\{pt\}, \mathbb{Z}) = 0$ but

$$H^1(\rho\{pt\}, \mathbb{Z}) = H^1(\chi(\mathbb{R}_+), \mathbb{Z}) = C(\mathbb{R}_+)/C_b(\mathbb{R}_+).$$

Of particular interest to us are the classifying spaces that are also Eilenberg–MacLane spaces. For the circle group S^1 we have $BS^1 = K(\mathbb{Z}, 2)$. For the symmetric group S_n on n letters we have $BS_n = K(S_n, 1)$. Explicit constructions of these spaces are well known: For S^1 the total space ES^1 of the universal principal S^1-bundle is S^∞ (the inductive limit of n-spheres) and $BS^1 = S^\infty/S^1 = \mathbb{C}P^\infty$ (the inductive limit of complex projective n-spaces). For S_n we have $ES_n = V_{n,\infty}$ (the inductive limit of Stiefel manifolds $V_{n,m}$ of orthonormal n-frames in $\mathbb{R}^m$ or $\mathbb{C}^m$) and $BS_n = V_{n,\infty}/S_n$. For these spaces, where the classifying groups are compact, 4.4 applies.

COROLLARY 4.7. *For a compact Hausdorff space X with real corona $\rho(X)$ the cohomology sets (resp. group)*

$$H^1(X, S_n) \qquad (\textit{resp. } H^2(X, \mathbb{Z}))$$

are trivial if and only if the corresponding sets (resp. group)

$$H^1(\rho(X), S_n) \qquad (resp.\ H^2(\rho(X), \mathbb{Z}))$$

are trivial.

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